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The problem of the reinforcement of a plate with a cutout by a two-dimensional patch☆

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Abstract

The problem of the reinforcement of an elastic plate with a cutout by means of a two-dimensional patch, which completely covers the cutout and is rigidly fixed to the plate along its boundary, is considered. The cutout and the patch can be of arbitrary shape. The problem is reduced to a system of three singular integral equations using of special integral representations which describe the stress state in the plate and in the patch. The unique solvability of the system is proved. Examples are presented.

This problem has been explicitly solved using power series and conformal mappings methods[1,2] for special cases when the cutout in the plate and the reinforcing patch are bounded by concentric circles or confocal ellipses. The related problem of the reinforcement of a plate with a circular cutout by means of a concentric or eccentric patch, which is joined to the plate not only along its boundary but also along the boundary of the cutout, has been solved using the same methods.[3,4] Methods of reinforcing a plate with a cutout using a two-dimensional patch which is glued to the plate along its surface have been studied using analytical and numerical methods.[5–7] A number of papers[8–13] have investigated the strength properties of a plate with cracks which have been reinforced by two-dimensional patches that are continuously fixed to the plate over their surfaces or discretely using rivets. Plates with cracks which are reinforced by two-dimensional patches, fixed to the plate rigidly or elastically along their boundaries, have been investigated in Refs. 14–16. A detailed review of the results concerning the above and other methods for the reinforcement and maintenance of plates with cracks and cutouts by means of patches (two- or one-dimensional) can be found in Refs. 11,17.

1. Formulation of the problem

Suppose a thin elastic patch S_0 is placed on an infinite thin elastic plate S with a cutout. This patch completely covers the cutout and is rigidly fixed to the plate along the whole of its boundary L_0 (Fig. 1). The plate and the patch are homogeneous, isotropic and have a thickness, shear modulus and Poisson's ratio h, μ and ν and h_0, μ_0 and ν_0 respectively. The contours l and L_0 are simple closed Lyapunov curves which do not share any common points. We go round the contour L clockwise and the contour L_0 anticlockwise. The origin of coordinates is located within L.

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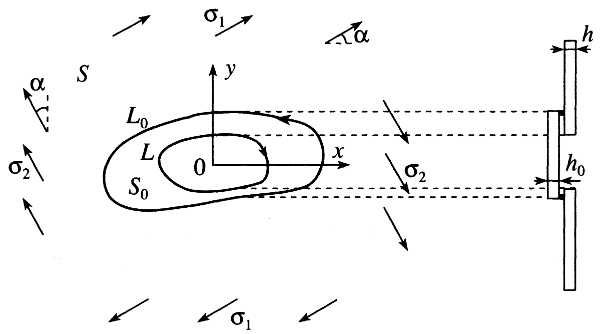

Fig. 1.

The principal tensile stresses σ_1 and σ_2, which are located in the plane of the plate and at an angle of α and $\alpha + \pi/2$ to the real axis respectively, act on the plate at infinity. Specified loads

$$(\sigma_n + i\tau_n)(t) = p(t), \quad t \in L \tag{1.1}$$

act on the boundary of the cutout, where σ_n and τ_n are the normal and tangential components of the vector of the external stresses acting on an area element which is tangential to the boundary of the cutout. Henceforth, in the calculations, all the stresses are taken per unit thickness of the plate or the patch.

The plate and the patch only interact with one another through the line which joins them, in which the rigid joining conditions:

$$\begin{aligned} &(u + iv)^+(t) = (u + iv)^-(t) = (u + iv)_0^+(t), \quad t \in L_0 \\ &h_0(\sigma_n + i\tau_n)_0^+(t) + h(\sigma_n + i\tau_n)^+(t) = h(\sigma_n + i\tau_n)^-(t), \quad t \in L_0 \end{aligned} \tag{1.2}$$

are satisfied, where $u + iv$ is the strain vector, the superscript plus (minus) corresponds to the limit value of some parameter or other to the left (to the right) of the contour, parameters with a zero subscript refer to the patch, and those without a zero subscript refer to the plate. The surfaces of the plate and the patch come into contact without friction.

We will assume that the plate and the patch are under conditions of generalized plane stress, which has also to be determined.

2. Uniqueness of the solution of the problem

The problem can be considered as the first fundamental problem in the theory of elasticity for a "plate - patch" structure. We shall show that, if a solution of this problem exists, then it is unique.

Assuming that the stresses and strains are continuous in the closures of the domains into which the above-mentioned structure is subdivided by the joining line L_0, we consider the integral

$$J = h\int_L (X_n u + Y_n v)^+ ds$$

where ds is an element of the length of an arc, and X_n^+ and Y_n^+ are the horizontal and vertical components of the stress vector acting in the plate on an area element which is tangential to the contour L. We now add to and subtract from J the analogous integral along the line L_0:

$$J = h\int_L (X_n u + Y_n v)^+ ds + h\int_{L_0} (X_n u + Y_n v)^+ ds - h\int_{L_0} (X_n u + Y_n v)^+ ds \tag{2.1}$$

It follows from the equality $X_n + iY_n = (\sigma_n + i\tau_n)e^{i\theta}$, where θ is the angle between the real axis and the right normal to the line L_0, and from the last condition of (1.2) that

$$h_0(X_n + iY_n)_0^+(t) + h(X_n + iY_n)^+(t) = h(X_n + iY_n)^-(t), \quad t \in L_0$$

and, therefore, taking account of the first two conditions of (1.2), we have

$$h\int_{L_0}(X_nu + Y_nv)^+ds = h\int_{L_0}(X_nu + Y_nv)^-ds - h_0\int_{L_0}(X_nu + Y_nv)_0^+ds \tag{2.2}$$

Replacing the last integral in (2.1) by the sum (2.2), we obtain

$$J = h\int_{\partial S_1}(X_nu + Y_nv)^+ds + h\int_{\partial S_2}(X_nu + Y_nv)^+ds + h_0\int_{\partial S_0}(X_nu + Y_nv)_0^+ds \tag{2.3}$$

where the integrals are taken along the boundary of the part S_1 of the plate located between the lines l and L_0, of the part S_2 of the plate located outside L_0 and along the patch boundary S_0. On the right-hand side of equality (2.3), the parameters X_n, Y_n, u, v refer to the plate, and the parameters $(X_n, Y_n, u, v)_0$ refer to the patch. If they refer to the difference between two solutions of the problem being considered, then $J = 0$. When $z \to \infty$, the stresses in the plate, corresponding to the difference in the solutions, decreases as $|z|^{-2}$ and Green's formula can therefore be applied to the integrals on the right-hand side of equality (2.3). As a result, using an approach analogous to that in Ref. 18, we obtain

$$J = h\iint_S(\lambda\theta^2 + 2\mu(e_{xx}^2 + 2e_{xy}^2 + e_{yy}^2))dxdy + h_0\iint_{S_0}(\lambda\theta^2 + 2\mu(e_{xx}^2 + 2e_{xy}^2 + e_{yy}^2))_0dxdy$$

where λ, μ and λ_0, μ_0 are the Lamé constants of the plate and the patch respectively, and $\theta = e_{xx} + e_{yy}$ and e_{xx}, e_{xy}, e_{yy} are the components of the strain tensor. Since the constants λ, μ, λ_0, μ_0 and are positive $J = 0$, it follows from the last equality that the strain tensor, and of course all the stresses are equal to zero everywhere in the plate and in the patch. Hence, if the problem has a solution, it is unique.

Remark 1. The second fundamental problem in the theory of elasticity for a "plate - patch" structure when, instead of the stresses, the displacements on the boundary L of the cutout are specified:

$$(u + iv)(t) = p(t), \quad t \in L \tag{2.4}$$

will also have unique solution. Problem (1.2), (2.4) corresponds to the reinforcement of a plate with a cutout by means of an elastic patch S_0 and an absolutely rigid thin laminated insert into the cutout which is rigidly fixed to the plate along the line L. In this case, for the problem to be uniquely solvable it is additionally necessary to specify the further principal force vector which acts on the boundary of the cutout that is equivalent to the application of a specified force to the rigid insert.

3. Integral representations of complex potentials

We will now formally supplement the plate S and the patch S_0 up to the complete planes such that they change continuously on passing across the boundary of the cutout in the first stress plane, and the strains outside the patch will also be zero in the second stress plane. Then, when account is taken of conditions (1.2), the complex potentials, in terms of which the stresses and strains in the plate and the patch are expressed, can be taken in the form[19]

$$\Phi(z) = \Gamma - \frac{Q}{z} + \frac{1}{2\pi}\int_L \frac{g'(t)dt}{t-z} + \frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0} \frac{q(t)dt}{t-z}$$

$$\Psi(z) = \Gamma' + \frac{\kappa\bar{Q}}{z} + \frac{M}{2\pi i z^2} + \frac{1}{2\pi}\int_L \left(\frac{\overline{g'(t)dt}}{t-z} - \frac{\bar{t}\,g'(t)dt}{(t-z)^2}\right) +$$

$$+ \frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0}\left(\frac{\kappa\overline{q(t)dt}}{t-z} - \frac{\bar{t}\,q(t)dt}{(t-z)^2}\right), \quad z \in S \tag{3.1}$$

$$\Phi_0(z) = \frac{1}{2\pi}\int_{L_0}\left(g_0'(t) + \frac{2ih_*^{-1}q(t)}{\kappa_0+1}\right)\frac{dt}{t-z}$$

$$\Psi_0(z) = \frac{1}{2\pi}\int_{L_0}\left[\left(\overline{g_0'(t)} + \frac{2ih_*^{-1}\kappa_0\overline{q(t)}}{\kappa_0+1}\right)\frac{\overline{dt}}{t-z} - \left(g_0'(t) + \frac{2ih_*^{-1}q(t)}{\kappa_0+1}\right)\frac{\bar{t}\,dt}{(t-z)^2}\right], \quad z \in S_0$$

where

$$\Gamma = \frac{\sigma_1+\sigma_2}{4}, \ \Gamma' = \frac{\sigma_2-\sigma_1}{2}e^{-2i\alpha}, \ Q = \frac{X+iY}{2\pi(1+\kappa)}, \ \kappa = \frac{3-\nu}{1+\nu}, \ \kappa_0 = \frac{3-\nu_0}{1+\nu_0}, \ h = \frac{h_0}{h} \tag{3.2}$$

$$X + iY = -i\int_L p(t)dt, \quad M = -\mathrm{Re}\int_L \bar{t}\,p(t)dt$$

$$g'(t) = \frac{2\mu}{i(\kappa+1)}\frac{d}{dt}((u+iv)^+(t) - (u+iv)^-(t)), \quad t \in L$$

$$q(t) = \frac{1}{2}((\sigma_n + i\tau_n)^+(t) - (\sigma_n + i\tau_n)^-(t)), \quad t \in L_0 \tag{3.3}$$

$$g_0'(t) = \frac{2\mu_0}{i(\kappa_0+1)}\frac{d}{dt}(u+iv)_0^+(t), \quad t \in L_0$$

Here, $X + iY$ and M are the principal vector and the principal moment (with respect to the origin of coordinates) of the external stresses acting on the boundary of the cutout and $g'(t)$, $q(t)$, $g_0'(t)$ are unknown functions which, like the specified function $p(t)$, we shall assume to be continuous in Hölder's sense on the corresponding lines.

In deriving the representations for $\Phi_0(z)$ and $\Psi_0(z)$, account has been taken of the fact that the discontinuity in the strain vector in the second auxiliary plane on crossing the line L_0 is equal to $(u + iv)_0^+(t)$ and the discontinuity in the expression $(\sigma_n + i\tau_n)_0$ is equal to $-2h_*^{-1}\,q(t)$ by virtue of the last equality of (1.2).

Expanding the functions $\Phi(z)$ and $\Psi(z)$ in Laurent series in the neighbourhood of infinity and comparing them with the well-known Muskhelishvili representations of complex potentials in the neighbourhood of infinity,[18] we obtain

$$\int_L g'(t)dt = 0, \quad \int_{L_0} q(t)dt = 0 \tag{3.4}$$

The first of conditions (3.4) expresses the uniqueness of the strains on going round the cutout in the plate and the second condition expresses the equilibrium of the "plate - patch" structure. To these conditions it is necessary to add the further condition for the strains in the patch to be unique

$$\int_{L_0} g_0'(t)dt = 0 \tag{3.5}$$

4. The integral equations of the problem

Suppose L' is a smooth orientated curve lying in the closure $\bar{S}$ of the domain S. In particular, L' can coincide with L or L_0. Then, the normal and tangential components of the stress vector acting on area element which is tangential to the curve L' as viewed from the right normal and the derivative of the strain vector $u + iv$ at the points of the curve L' are found in terms of the potentials $\Phi(z)$ and $\Psi(z)$ using the formulae[18,19]

$$\begin{aligned}(\sigma_n + i\tau_n)(t) &= \Phi(t) + \overline{\Phi(t)} + \frac{d\bar{t}}{dt}(t\overline{\Phi'(t)} + \overline{\Psi(t)})\\ 2\mu\frac{d}{dt}(u+iv)(t) &= \kappa\Phi(t) - \overline{\Phi(t)} - \frac{d\bar{t}}{dt}(t\overline{\Phi'(t)} + \overline{\Psi(t)}), \quad t \in L'\end{aligned} \tag{4.1}$$

These formulae also hold at points of the curve located in the closure $\bar{S}_0$ of the patch, if all the parameters in them are taken with a zero subscript.

On the basis of formulae (4.1) and representations (3.1), on satisfying conditions (1.1) and (1.2), we obtain a system of three singular integral equations in the closed contours L and L_0 for finding the three unknown functions $g'(t)$, $t \in L$ and $q(t)$, $g_0'(t)$, $t \in L_0$

$$\begin{aligned}
&\frac{1}{2\pi}\int_L\left(\frac{1}{\tau - t} + \frac{1}{\bar{\tau} - \bar{t}}\frac{d\bar{t}}{dt}\right)g'(\tau)d\tau + \frac{1}{2\pi}\int_L\left(\frac{1}{\bar{\tau} - \bar{t}} - \frac{\tau - t}{(\bar{\tau} - \bar{t})^2}\frac{d\bar{t}}{dt}\right)\overline{g'(\tau)}\overline{d\tau} +\\
&+\frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0}\left(\frac{1}{\tau - t} - \frac{\kappa}{\bar{\tau} - \bar{t}}\frac{d\bar{t}}{dt}\right)q(\tau)d\tau - \frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0}\left(\frac{1}{\bar{\tau} - \bar{t}} - \frac{\tau - t}{(\bar{\tau} - \bar{t})^2}\frac{d\bar{t}}{dt}\right)\overline{q(\tau)d\tau} +\\
&+\left(\frac{1}{\pi}\int_L g'(\tau)d\tau\right)\frac{|dt|}{dt} = p_1(t), \quad t \in L\\
&\mu_*\left\{p_2(t) + \frac{1}{2\pi}\int_L\left(\frac{\kappa}{\tau - t} - \frac{1}{\bar{\tau} - \bar{t}}\frac{d\bar{t}}{dt}\right)g'(\tau)d\tau - \frac{1}{2\pi}\int_L\left(\frac{1}{\bar{\tau} - \bar{t}} - \frac{\tau - t}{(\bar{\tau} - \bar{t})^2}\frac{d\bar{t}}{dt}\right)\overline{g'(\tau)}\overline{d\tau} +\right.\\
&+\frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0}\left(\frac{\kappa}{\tau - t} + \frac{\kappa}{\bar{\tau} - \bar{t}}\frac{d\bar{t}}{dt}\right)q(\tau)d\tau + \frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0}\left(\frac{1}{\bar{\tau} - \bar{t}} - \frac{\tau - t}{(\bar{\tau} - \bar{t})^2}\frac{d\bar{t}}{dt}\right)\overline{q(\tau)d\tau} +\\
&\left.+\left(\frac{1}{\pi}\int_L q(\tau)d\tau\right)\frac{|dt|}{dt}\right\} = i(\kappa_0 + 1)g_0'(t), \quad t \in L_0\\
&-\frac{i(\kappa_0 + 1)g_0'(t)}{2} + \frac{1}{2\pi}\int_{L_0}\left(\frac{\kappa_0}{\tau - t} + \frac{1}{\bar{\tau} - \bar{t}}\frac{d\bar{t}}{dt}\right)g_0'(\tau)d\tau +\\
&+\frac{1}{2\pi}\int_{L_0}\left(\frac{-1}{\bar{\tau} - \bar{t}} + \frac{\tau - t}{(\bar{\tau} - \bar{t})^2}\frac{d\bar{t}}{dt}\right)\overline{g_0'(\tau)}\overline{d\tau} + \frac{ih_*^{-1}}{\pi(\kappa_0 + 1)}\int_{L_0}\left(\frac{\kappa_0}{\tau - t} + \frac{\kappa_0}{\bar{\tau} - \bar{t}}\frac{d\bar{t}}{dt}\right)q(\tau)d\tau +\\
&+\frac{ih_*^{-1}}{\pi(\kappa_0 + 1)}\int_{L_0}\left(\frac{1}{\bar{\tau} - \bar{t}} - \frac{\tau - t}{(\bar{\tau} - \bar{t})^2}\frac{d\bar{t}}{dt}\right)\overline{q(\tau)d\tau} = 0, \quad t \in L_0
\end{aligned} \tag{4.2}$$

where

$$\mu_* = \frac{\mu_0}{\mu}, \quad p_1(t) = -2\mathrm{Re}\Gamma + 2\mathrm{Re}\frac{Q}{t} - \frac{\overline{dt}}{dt}\left(\bar{\Gamma}' + \frac{\bar{Q}t}{\bar{t}^2} + \frac{\kappa Q}{\bar{t}} - \frac{M}{2\pi i\bar{t}^2}\right) + p(t)$$

$$p_2(t) = \kappa\Gamma - \bar{\Gamma} - \bar{\Gamma}'\frac{\overline{dt}}{dt} - \kappa Q\left(\frac{1}{t} + \frac{1}{\bar{t}}\frac{\overline{dt}}{dt}\right) + \bar{Q}\left(\frac{1}{t} - \frac{1}{\bar{t}^2}\frac{\overline{dt}}{dt}\right) + \frac{M}{2\pi i\bar{t}^2}\frac{\overline{dt}}{dt}$$

For convenience in deriving system (4.2), the functionals $\left(\frac{1}{\pi}\int_L g'(t)d\tau\right)\frac{|dt|}{dt}$ and $\left(\frac{1}{\pi}\int_{L_0} q(\tau)d\tau\right)\frac{|dt|}{dt}$, which are equal to zero, are added to the left-hand sides of the first and second equations respectively.

We will now show that any solution of system (4.2), if such a solution exists, satisfies conditions (3.4) and (3.5).

Using the notation

$$a = \frac{1}{\pi}\int_L g'(\tau)d\tau, \quad b = \frac{1}{\pi}\int_{L_0} q(\tau)d\tau$$

and taking account of the representations (3.1), we can write system (4.2) in the form

$$\Phi(t) + \overline{\Phi(t)} + \frac{\overline{dt}}{dt}(t\overline{\Phi'(t)} + \overline{\Psi(t)}) + a\frac{|dt|}{dt} = p(t), \quad t \in L$$

$$\mu_*\left(\kappa\Phi(t) - \overline{\Phi(t)} - \frac{\overline{dt}}{dt}(t\overline{\Phi'(t)} + \overline{\Psi(t)}) + b\frac{|dt|}{dt}\right) = \tag{4.3}$$

$$= \left(\kappa_0\Phi_0(t) - \overline{\Phi_0(t)} - \frac{\overline{dt}}{dt}(t\overline{\Phi_0'(t)} + \overline{\Psi_0(t)})\right)^+ = i(\kappa_0 + 1)g_0'(t), \quad t \in L_0$$

where $\Phi(t)$, $\Psi(t)$, $\Phi_0(t)$, $\Psi_0(t)$ are the direct (principal) values of the functions (integrals) (3.1) on the lines L_0, and $\Phi^+(t)$, $\Psi^+(t)$, $\Phi_0^+(t)$, $\Psi_0^+(t)$ are their limit values from the left on L_0. On the line L, the direct values of the functions $\Phi(z)$, $\bar{z}\Phi'(z) + \Psi(z)$ and their limit values from the right (from within the contour formed by the line L) are related by the equalities[19]

$$\Phi^-(t) = -\frac{i}{2}g'(t) + \Phi(t), \quad (\bar{t}\Phi'(t) + \Psi(t))^- = \frac{i}{2}(g'(t) - \overline{g'(t)})\frac{\overline{dt}}{dt} + \bar{t}\Phi'(t) + \Psi(t)$$

on the basis of which we can write the first equation of (4.3) in the form

$$\Phi^-(t) + \overline{\Phi^-(t)} + \frac{\overline{dt}}{dt}(t\overline{\Phi'(t)} + \overline{\Psi(t)})^- + a\frac{|dt|}{dt} = p(t), \quad t \in L$$

and integrate along the line L, treating it (after changing the direction of passing round it) as the boundary of a simply-connected domain which coincides with the cutout. Since the functions $\Phi(z)$, $\Psi(z)$ are continuous on the contour L and analytic within it, apart from at the point $z = 0$, where they have poles with residues $-Q$ and $\kappa\bar{Q}$ respectively, then

$$\int_L \Phi^-(t)dt = 2\pi iQ, \quad \int_L \frac{\overline{dt}}{dt}\overline{\Psi^-(t)}dt = \overline{\int_L \Psi^-(t)dt} = 2\pi i\kappa Q$$

$$\int_L \left(\overline{\Phi(t)} + \frac{\overline{dt}}{dt}t\overline{\Phi'(t)}\right)^- dt = \int_L d(t\overline{\Phi^-(t)}) = 0$$

After integration, we thereby obtain

$$2\pi i(\kappa+1)Q + a\int\limits_L |dt| = \int\limits_L p(t)dt$$

where

$$2\pi i(\kappa+1)Q = i(X+iY), \quad \int\limits_L p(t)dt = i(X+iY)$$

Consequently, $a = 0$, which is equivalent to satisfying the first condition of (3.4). On carrying out similar actions with the second equation of (4.3) on the line L_0, we obtain the second condition of (3.4) and, from the last equation of (4.3) by virtue of the analyticity of the functions $\Phi_0(z)$, $\Psi_0(z)$ within the contour L_0, we obtain condition (3.5).

Hence, for the solution of the problem we have the system of singular integral equations in the class of Hölder functions.

Remark 2. When $\mu_0 = 0$. Which corresponds to the case when there is no patch, it follows from the second equation of system (4.2) that $g_0'(t) = 0$, $t \in L_0$. The third equatio n of the system then becomes a homogeneous integral equation for the second fundamental problem in the theory of elasticity for a patch S_0 and it therefore[19] only has the trivial solution $q(t) = 0$, $t \in L_0$. The same equation in $q(t)$ likewise also holds when $h_0 = 0$. As a result, the first equation of the system is transformed into the equation of the first fundamental problem of the theory of elasticity for an infinite plate S with a patch.

Remark 3. The representations (3.1) and the system of Eq. (4.2) also do not change their form when the patch covers several cutouts which are bounded by closed Lyapunov curves $L_1, L_2, \ldots, L_n$ if, by L, we mean the set of these curves. In this case, instead of the terms

$$-\frac{Q}{z} \text{ and } \frac{\kappa\bar{Q}}{z} + \frac{M}{2\pi i z^2}$$

it is necessary to take the terms

$$-\sum_{k=1}^{n} \frac{Q_k}{z-z_k} \text{ and } \sum_{k=1}^{n}\left(\frac{\kappa\bar{Q}_k}{z-z_k} + \frac{M_k}{2\pi i(z-z_k)^2}\right)$$

in the representations for $\Phi(z)$ and $\Psi(z)$ respectively, where z_k is an arbitrary fixed point within a contour L_k and

$$Q_k = (2\pi(\kappa+1))^{-1}(X_k+iY_k), \quad X_k+iY_k = -i\int\limits_{L_k} p(t)dt, \quad M_k = -\mathrm{Re}\int\limits_{L_k} \bar{t}p(t)dt$$

It is clear that, in this case, the first equation of the system decomposes into n equations on the curves L_k for each of which it is necessary to take "its own" functional $\left(\frac{1}{\pi}\int_{L_k} g'(\tau)d\tau\right)\frac{|dt|}{dt}$ instead of the functional $\left(\frac{1}{\pi}\int_L g'(\tau)d\tau\right)\frac{|dt|}{dt}$.

5. The unique solvability of the system of integral equations

We will transform system (4.2) to the form

$$
\begin{gathered}
\frac{1}{\pi}\int_L \frac{g'(\tau)d\tau}{\tau-t} = -\frac{1}{2\pi}\int_L\left(M_1(\tau,t)+2\frac{|dt|}{dt}\right)g'(\tau)d\tau-\frac{1}{2\pi}\int_L M_2(\tau,t)\overline{g'(\tau)d\tau}+\\
+\frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0}\left(\frac{\kappa-1}{\tau-t}+\kappa M_1(\tau,t)\right)q(\tau)d\tau+\frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0} M_2(\tau,t)\overline{q(\tau)d\tau}+p_1(t),\quad t\in L\\
-\frac{i(\kappa_0+1)}{2}\left(1+\frac{2h_*^{-1}\mu_*^{-1}\kappa_0(\kappa+1)}{\kappa(\kappa_0+1)}\right)g_0'(t)+\frac{\kappa_0-1}{2\pi}\int_{L_0}\frac{g_0'(\tau)d\tau}{\tau-t} =\\
= -\frac{\beta}{2\pi}\int_L\left(\frac{\kappa-1}{\tau-t}-M_1(\tau,t)\right)g'(\tau)d\tau+\frac{\beta}{2\pi}\int_L M_2(\tau,t)\overline{g'(\tau)d\tau}-\frac{\beta}{\pi}\frac{|dt|}{dt}\int_{L_0}q(\tau)d\tau-\\
-\frac{h_*^{-1}(\kappa_0-\kappa)}{\kappa(\kappa_0+1)}\frac{1}{\pi i}\int_{L_0}M_2(\tau,t)\overline{q(\tau)d\tau}+\frac{1}{2\pi}\int_{L_0}M_1(\tau,t)g_0'(\tau)d\tau+\\
+\frac{1}{2\pi}\int_{L_0}M_2(\tau,t)\overline{g_0'(\tau)d\tau}-\beta p_2(t),\quad t\in L_0\\
\frac{1}{\pi i}\int_{L_0}\left(\left(\frac{2\kappa}{\kappa+1}+\frac{4h_*^{-1}\mu_*^{-1}\kappa_0}{\kappa_0+1}\right)q(\tau)-i\mu_*^{-1}(\kappa_0-1)g_0'(\tau)\right)\frac{d\tau}{\tau-t} =\\
= -\frac{1}{2\pi}\int_L\left(\frac{\kappa-1}{\tau-t}-M_1(\tau,t)\right)g'(\tau)d\tau+\frac{1}{2\pi}\int_L M_2(\tau,t)\overline{g'(\tau)d\tau}-\\
-\frac{1}{\pi i}\int_{L_0}\left(\left(\frac{\kappa}{\kappa+1}+\frac{2h_*^{-1}\mu_*^{-1}\kappa_0}{\kappa_0+1}\right)M_1(\tau,t)+i\frac{|dt|}{dt}\right)q(\tau)d\tau-\\
-\left(\frac{1}{\kappa+1}+\frac{2h_*^{-1}\mu_*^{-1}}{\kappa_0+1}\right)\frac{1}{\pi i}\int_{L_0}M_2(\tau,t)\overline{q(\tau)d\tau}-\frac{\mu_*^{-1}}{\pi}\int_{L_0}M_1(\tau,t)g_0'(\tau)d\tau-\\
-\frac{\mu_*^{-1}}{\pi}\int_{L_0}M_2(\tau,t)\overline{g_0'(\tau)d\tau}-p_2(t),\quad t\in L_0
\end{gathered}
\tag{5.1}
$$

where

$$
\beta = \frac{h_*^{-1}\kappa_0(\kappa+1)}{\kappa(\kappa_0+1)},\quad M_1(\tau,t) = \frac{d}{dt}\ln\frac{\tau-t}{\bar\tau-\bar t},\quad M_2(\tau,t) = -\frac{d}{dt}\frac{\tau-t}{\bar\tau-\bar t}
$$

Only regular terms occur on the right-hand sides of the equations of system (5.1) while their left-hand sides are characteristic parts of the equations in the individual functions

$$f_1(t) = g'(t), \quad t \in L$$
$$f_2(t) = g_0'(t), \quad f_3(t) = \left(\frac{2\kappa}{\kappa+1} + \frac{4h_*^{-1}\mu_*^{-1}\kappa_0}{\kappa_0+1}\right)q(t) - \frac{i(\kappa_0-1)}{\mu_*}g_0'(t), \quad t \in L_0 \tag{5.2}$$

All these equations have a zero index and, hence,[20] they allow of equivalent regularization, for example, by the Carleman-Vekua method. As a result of using this method, an equivalent system of three Fredholm integral equations of the second kind of the form

$$\begin{aligned} f_1(t) &= \int_L (m_{11}(\tau,t)f_1(\tau)d\tau + n_{11}(\tau,t)\overline{f_1(\tau)d\tau}) + \\ &+ \sum_{j=2}^{3}\int_{L_0}(m_{1j}(\tau,t)f_j(\tau)d\tau + n_{1j}(\tau,t)\overline{f_j(\tau)d\tau}) + r_1(t), \quad t \in L \\ f_k(t) &= \int_L (m_{k1}(\tau,t)f_1(\tau)d\tau + n_{k1}(\tau,t)\overline{f_1(\tau)d\tau}) + \\ &+ \sum_{j=2}^{3}\int_{L_0}(m_{kj}(\tau,t)f_j(\tau)d\tau + n_{kj}(\tau,t)\overline{f_j(\tau)d\tau}) + r_k(t), \quad t \in L_0, \quad k = 2, 3 \end{aligned} \tag{5.3}$$

is obtained, where $m_{kj}(\tau, t)$, $n_{kj}(\tau, t)$ are regular kernels, which are expressed in terms of the kernels $M_1(\tau, t)$, $M_2(\tau, t)$ and integrals of them. The form of the kernels is not important for the subsequent reasoning, and they are not presented here on account of their length. The functions $r_k(t)$ are expressed in terms of the functions $p_1(t)$, $p_2(t)$ and particular integrals of them, and, therefore, like the last functions, are continuous in the Hölder sense.

System (5.3) contains both the unknown functions $f_k(t)$ $(k = 1, 2, 3)$ as well as the complex conjugates to them. Taking the complex conjugate of Eq. (5.3) and introducing the new functions

$$f_k(t) = \overline{f_{k-3}(t)}, \quad k = 4, 5, 6 \tag{5.4}$$

we obtain a system of six ordinary Fredholm integral equations in the six unknown functions $f_k(t)$ $(k = 1, 2, \ldots, 6)$ under the additional conditions $f_k(t) = \overline{f_{k+3}(t)}$ $(k = 1, 2, 3)$. These conditions can always be satisfied. In fact, we will assume that there is a certain solution $f_k^*(t)$ $(k = 1, 2, \ldots, 6)$ of the system. Then, $f_4^*(t)$, $f_5^*(t)$, $f_6^*(t)$, $f_1^*(t)$, $f_2^*(t)$, $f_3^*(t)$ will likewise be a solution of the system, and, hence, the half sum of these two solutions will be a solution of the system which satisfies the above-mentioned additional conditions.

We will now show that the homogeneous system of Fredholm equation, corresponding to system (5.3), only has a trivial solution. By virtue of the equivalence of systems (4.2) and (5.3), this homogeneous system is a consequence of the "homogeneous" mechanical problem when there are no stresses in the plate on the boundary of the cutout and at infinity. Then, by virtue of the uniqueness of the solution of the problem, all the stresses in the plate and the patch are equal to zero, and, hence,[18] the complex potentials $\Psi(z)$ and $\Psi_0(z)$, describing the stress state are identically equal to zero, and the complex potentials $\Phi(z)$ and $\Phi_0(z)$ are determined, apart from imaginary terms: $\Phi(z) = i\gamma_j$, $z \in S_j (j = 1, 2)$; $\Phi_0(z) = i\gamma_0$, $z \in S_0$, where all the γ_j are real. Also, $\Psi(z) = 0$, $\Phi(z) = i\gamma_3$ in the complement $S_3 = C\backslash S$ of the plate up to the auxiliary plane since, on passing across the line L from S to S_3, the stresses change continuously, and they are therefore equal to zero on the boundary of the domain S_3, and this means that they are also zero in the whole of the domain. Consequently, by formulae (3.3) and (4.1),

$$g'(t) = \gamma_1 - \gamma_3, \quad t \in L; \quad q(t) = 0, \quad g_0'(t) = \gamma_0, \quad t \in L_0$$

We will now calculate the principal moment (with respect to the origin of coordinates) of the forces acting on the line L from the right. On the one hand, $M = 0$. On the other hand,[19]

$$M = -2\mathrm{Im}\int_L \bar{t}\, g'(t)dt = 2(\gamma_1 - \gamma_3)\int_L (ydx - xdy) = 4(\gamma_3 - \gamma_1)\iint_{S_3} dxdy$$

where the double integral is equal to the area of the domain S_3. Hence, $M = 0$ if and only if $\gamma_3 - \gamma_1 = 0$. Consequently, $g'(t) = 0$, $t \in L$. Similarly, on calculating the principal moment of the forces acting on the line L_0 from the right, we obtain $g_0'(t) = 0$, $t \in L_0$. All the functions (5.2) and (5.4) are then equal to zero and the homogeneous system of Fredholm equations obtained from (5.3) only has a trivial solution, and the corresponding inhomogeneous system (5.3) is solvable for any Hölder functions $r_k(t)$.[20] Since the system of singular integral Eq. (4.2) is equivalent to the system of Fredholm Eq. (5.3), system (4.2) is solvable in the class of Hölder functions and, moreover, it is uniquely solvable by virtue of the uniqueness of the solution of the mechanical problem.

Remark 4. It is also easy to obtain the integral equations of the second fundamental problem (1.2), (2.4) for a "plate - patch" structure on the basis of representations (3.1). The first of these equations has the form

$$\frac{i(\kappa+1)}{2}g'(t) + \frac{1}{2\pi}\int_L \left(\frac{\kappa}{\tau - t} - \frac{1}{\bar{\tau} - \bar{t}}\frac{\overline{dt}}{dt}\right)g'(\tau)d\tau -$$

$$-\frac{1}{2\pi}\int_L \left(\frac{1}{\bar{\tau} - \bar{t}} - \frac{\tau - t}{(\bar{\tau} - \bar{t})^2}\frac{\overline{dt}}{dt}\right)\overline{g'(\tau)d\tau} + \frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0}\left(\frac{\kappa}{\tau - t} + \frac{\kappa}{\bar{\tau} - \bar{t}}\frac{\overline{dt}}{dt}\right)q(\tau)d\tau +$$

$$+\frac{(\kappa+1)^{-1}}{\pi i}\int_{L_0}\left(\frac{1}{\bar{\tau} - \bar{t}} - \frac{\tau - t}{(\bar{\tau} - \bar{t})^2}\frac{\overline{dt}}{dt}\right)\overline{q(\tau)d\tau} = 2\mu p'(t) - p_2(t), \quad t \in L_0$$

and the second and third equations are identical to the corresponding equations of system (4.2). The unique solvability of this last system is proved using the scheme described above.

6. Numerical calculations

Numerical calculations can be carried out both using the system of singular integral Eq. (4.2) as well as the equivalent system of Fredholm integral equations of the second kind, which is obtained from it. However, the Fredholm equations are less convenient to use on account of the complexity of their kernels. The method of mechanical quadrature[21] has been used for the numerical solution of system (4.2). Here, the formulae

$$\begin{gathered}\int_0^{2\pi} f(\phi)M(\phi, \theta)d\phi = \frac{2\pi}{n}\sum_{j=1}^{n} f(\phi_j)M(\phi_j, \theta)\\ f(\phi) = \frac{1}{n}\sum_{j=1}^{n} f(\phi_j)\sin\frac{n(\phi - \phi_j)}{2}\mathrm{ctg}\frac{\phi - \phi_j}{2}, \quad 0 \le \phi \le 2\pi, \quad \phi_j = \phi_0 + 2\pi\frac{j}{n}, \quad j = 1, 2, \ldots, n\end{gathered} \tag{6.1}$$

were used for the approximation of the integrals and the unknown functions, where n is an even natural number, ϕ_0 is an arbitrarily fixed real number and ϕ is a real parameter, which is used to specify the boundaries of the cutout and the patch, L: $t = \omega(\phi)$ and L_0: $t = \omega_0(\phi)$, $\phi \in [0, 2\pi]$. The first formula of (6.1) is true for any $\theta \in [0, 2\pi]$ if the kernel $M(\phi, \theta)$ is regular and, when $\theta = \theta_k = \phi_0 + \pi(2k - 1)/n$ $(k = 1, 2, \ldots, n)$, if $M(\phi, \theta)$ is a singular kernel.

Graphs of the stresses on the boundary of the cutout L and on the joining line L_0 are shown in Fig. 3 as a function of the polar angle θ (the polar axis originates at the centre of the square and is directed horizontally from left to right) in the case of a plate with a cutout in the form of a square with rounded corners, which is reinforced by a triangular patch

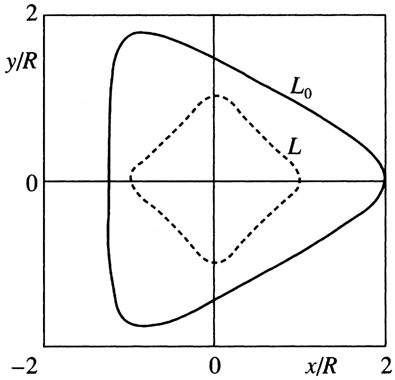

Fig. 2.

with rounded corners (Fig. 2). The boundaries of the cutout and the patch are specified by the parametric equations

$$L: t = \omega(\phi) = \frac{R+r}{2}e^{i\phi} + \frac{R-r}{2}e^{-3i\phi}, \quad r = 0.744R, \quad \phi \in [0, 2\pi]$$

$$L_0: t = \omega_0(\phi) = \frac{R_0+r_0}{2}e^{i\phi} + \frac{R_0-r_0}{2}e^{-2i\phi}, \quad R_0 = 2R, \quad r_0 = 1.25R, \quad \phi \in [0, 2\pi]$$

where r, R and r_0, R_0 are the radii of the circles inscribed in and circumscribed about the square L and the triangle L_0 respectively. The plate and the patch have the same thickness $h = h_0$, and the elastic parameters are $\mu = 73$ MPa, $\nu = 0.42$ and $\mu_0 = 40$ MPa, $\nu_0 = 0.37$ respectively. The boundary of the cutout is stress-free and a tensile stress $\sigma_1 = \sigma$ acts on the plate at infinity at an angle $\alpha = \pi/4$ to the positive direction of the real axis. The solid curve in Fig. 3*a* corresponds to the stress σ_s which acts on the normal cross-section to the boundary of the cutout L when there is a patch, and the dot-dash curve corresponds to the same stress when there is no patch. Graphs of the stresses in the plate inside and outside the contour L_0 are denoted by the numbers 1 and 2 in Fig. 3, *b*, *c* and *d* and, as viewed from the patch, by the number 3.

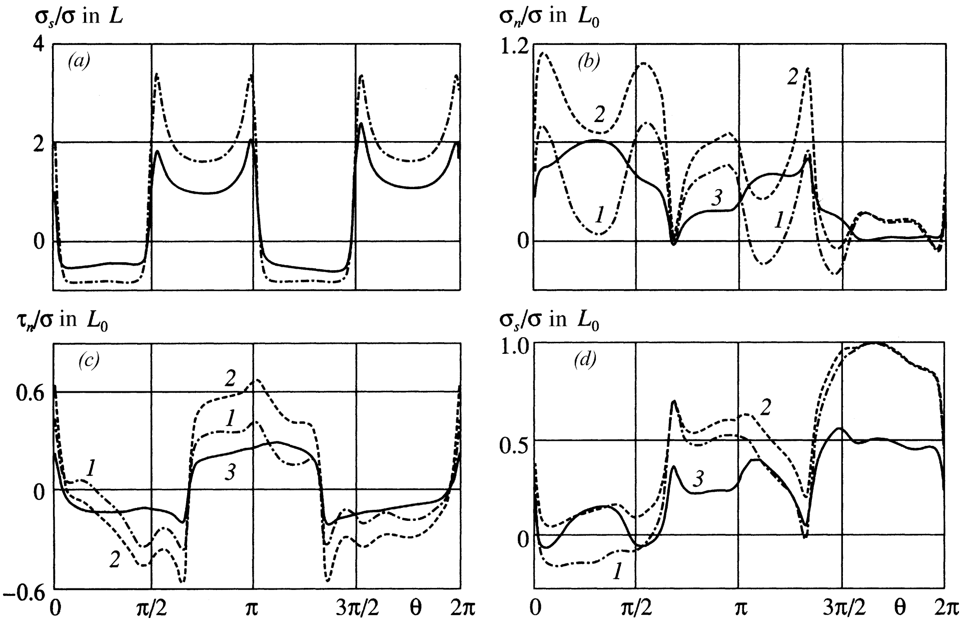

Fig. 3.

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